

QUASI RIGHT-VEERING BRAIDS AND NON-LOOSE LINKS

TETSUYA ITO AND KEIKO KAWAMURO

ABSTRACT. We introduce a notion of “quasi right-veering” for closed braids, which plays an analogous role to “right-veering” for abstract open books. We show that a transverse link is non-loose if and only if every closed braid representative with respect to an arbitrary open book decomposition of the underlying contact 3-manifold is quasi-right veering. We also show that several definitions for a “right-veering” closed braid are equivalent.

1. INTRODUCTION

The dichotomy between tight and overtwisted is fundamental to 3-dimensional contact topology and detecting tightness of a given contact structure often arises as an important problem.

In the context of Legendrian and transverse links in (overtwisted) contact 3-manifolds *non-loose* vs. *loose* dichotomy plays a role similar to the tight vs. overtwisted dichotomy. For instance, overtwisted contact structures are classified by homotopy equivalence [6], on the other hand loose null-homologous Legendrian (resp. transverse) links are coarsely classified (i.e. classified up to contactomorphism, smoothly isotopic to the identity) by the classical invariants, namely the Thurston-Bennequin number and the rotation number (resp. the self-linking number) [7, 9].

With the Giroux correspondence [14] between contact 3-manifolds and open books, Honda, Kazez and Matić [15, 16] show that checking right-veeringness of a mapping class gives an effective way to detect tightness of the compatible contact structure.

Theorem 1.1. [15, Theorem 1.1] *A contact 3-manifold (M, ξ) is tight if and only if for an arbitrary open book decomposition (S, ϕ) of (M, ξ) , ϕ is right-veering.*

Honda, Kazez and Matić also define the *fractional Dehn twist coefficient (FDTC)* [15]. It is an invariant of an open book and measures right-veering-ness of the monodromy. Hence the FDTC nicely reflects the type (tight or overtwisted) of its compatible contact structure [5, 16, 21].

As a natural counterpart of right-veering mapping classes, *right-veering closed braids* (in the open book (D^2, id) or in a general open book) have been defined and studied in the literature [2, 3, 25]. As a counterpart of the FDTC, in [19] we naturally extend it to the *FDTC for a closed braid L in an open book (S, ϕ)* with respect to a boundary component C of S , denote by $c(\phi, L, C)$, see also Definition 2.2. We demonstrate that various results on open books and the FDTC can be translated to results on closed braids and the FDTC for closed braids [19]. This gives us some hope that open books and closed braids in open books can be treated in a unified manner.

However this is too optimistic. Note that a non-right-veering open book supports an overtwisted contact structure [15], but a non-right-veering closed braid is not always loose. A simple example of this fact is a non-right-veering closed braid in an open book decomposition of a tight contact 3-manifolds.

In this paper, we find a condition on closed braids that guarantees looseness. in Section 3 we introduce *quasi-right-veering* closed braids. After studying basic properties of quasi-right-veering braids we show that it is the quasi-right-veering condition on closed braids that plays the same

role as the right-veering condition on open books in Theorem 1.1. Our first main result is the following and is proved in Section 4:

Theorem 1.2. *A transverse link K in a contact 3-manifold (M, ξ) is non-loose if and only if every closed braid representative of K with respect to an arbitrary open book decomposition that supports (M, ξ) is quasi right-veering.*

In Theorem 1.2 we allow the transverse link K to be empty. Our definition of quasi-right-veering implies that the empty braid with respect to an open book (S, ϕ) is quasi-right-veering if and only if ϕ is right-veering. Having a loose empty link can be interpreted as having an overtwisted underlying contact structure. Therefore we have Theorem 1.1 as a corollary of Theorem 1.2.

In Sections 5 and 6 we present more results concerning non-loose links.

The invariant *depth* is a measurement of non-looseness introduced by Baker and Onaran [1]. In Theorem 5.2 we relate depth-one links and non-quasi-right-veering braids.

Theorem 5.2. *Let K be a transverse link in (M, ξ) . Let (S, ϕ) be an open book supporting (M, ξ) . Suppose that $K = B \cup L$ the union of the binding, B , of (S, ϕ) and a closed braid L with respect to (S, ϕ) . Then the depth $d(K) = 1$ if and only if the braid L is non-quasi-right-veering.*

Theorem 6.1 below can be seen as a generalization of a corresponding result with regard to open books in [21].

Theorem 6.1. *Let L be a closed braid with respect to a planar open book (S, ϕ) . If $c(\phi, L, C) > 1$ for every boundary component C of S then L is non-loose.*

Finally in Section 7 we address one subtle but important issue on right-veering closed braids. As mentioned above, a couple of different looking definitions of right-veering closed braids have been existing in the literature, which we call ∂ -($\partial + P$), ∂ - ∂ , and ∂ - P right-veering (see Definition 7.2). We show that they are essentially equivalent (though there are subtle differences).

Corollary 7.8. *For $\psi \in \mathcal{MCG}(S; P)$ the following are equivalent.*

- (1) ψ is ∂ -($\partial + P$) right-veering.
- (2) ψ is ∂ - ∂ right-veering.
- (3) ψ is ∂ - P right-veering.

2. THE FRACTIONAL DEHN TWIST COEFFICIENTS OF CLOSED BRAIDS AND BRANCHED COVERINGS

Let $S \simeq S_{g,d}$ be an oriented compact surface with genus g and d boundary components. Throughout the paper we assume $d > 0$. Let $P = \{p_1, \dots, p_n\}$ be a (possibly empty) finite set of interior points of S . Let $\mathcal{MCG}(S; P)$ (denoted by $\mathcal{MCG}(S)$ if P is empty) be the mapping class group of the punctured surface $S \setminus P$, which is the group of isotopy classes of orientation preserving homeomorphisms of the surface S fixing ∂S pointwise and fixing P set-wise. By abuse of the notation, $\phi \in \mathcal{MCG}(S; P)$ often means a homeomorphism of S representing ϕ . Denote the *fractional Dehn twist coefficient (FDTC)* [15] with respect a boundary component C by $c(-, C) : \mathcal{MCG}(S; P) \rightarrow \mathbb{Q}$.

2.1. The FDTC of a braid. Let (S, ϕ) be an abstract open book supporting an oriented closed contact 3-manifold $(M, \xi) \cong (M_{(S, \phi)}, \xi_{(S, \phi)})$. Let L be a possibly empty closed n -braid with respect to the open book (S, ϕ) . Choose a base page S_0 of the open book. For each boundary component C of S we fix a collar neighborhood $\nu(C)$ of C and always assume that $\phi = id$ on $\nu(C)$. With a

transverse isotopy we may arrange L so that $L \cap S_0 \subset \nu(C)$. Put $P := L \cap S_0 \subset S_0 \cong S$. Cutting the 3-manifold M along the closure of the page S_0 we get a product region $S \times [0, 1]$ and the closed braid L gives rise to an n -braid, $\beta_L \subset S \times [0, 1]$. We regard β_L as an element of the n -stranded surface braid group $B_n(S)$.

Let $j : S \cong S \setminus \nu(C) \hookrightarrow S$ be an inclusion map that factors through $S \setminus \nu(C)$. The map j induces a map $j_* : \mathcal{MCG}(S) \rightarrow \mathcal{MCG}(S; P)$ such that $j_*(\phi) = id$ on $\nu(C)$. Recall the generalized Birman exact sequence [12, Theorem 9.1]

$$(2.1) \quad 1 \rightarrow B_n(S) \xrightarrow{i} \mathcal{MCG}(S; P) \xrightarrow{f} \mathcal{MCG}(S) \rightarrow 1$$

where i is the push map and f is the forgetful map.

Definition 2.1. Let L be a closed braid in (S, ϕ) with $P = L \cap S_0 \subset \nu(C)$. We call the mapping class

$$\phi_L := i(\beta_L) \circ j_*(\phi) \in \mathcal{MCG}(S; P)$$

the *distinguished monodromy* of the closed braid L .

We have

$$(2.2) \quad (M_{(S, \phi)}, L) \simeq ((S, P) \times [0, 1]) / \sim_{\phi_L}$$

where the equivalence relation “ \sim_{ϕ_L} ” satisfies $(x, 1) \sim (\phi_L(x), 0)$ for $x \in S$ and $(x, 1) \sim (x, t)$ for $x \in \partial S$.

Definition 2.2. We define the *fractional Dehn twist coefficient (FDTC)* of L with respect to C as the FDTC of the distinguished monodromy with respect to C

$$c(\phi, L, C) := c(\phi_L, C).$$

We have $c(\psi \circ \phi_L \circ \psi^{-1}, C) = c(\phi_L, C)$ for any $\psi \in \mathcal{MCG}(S; P)$. More strongly, we have the following:

Proposition 2.3. Let L_C and $L_{C'}$ be closed n -braids in the open book (S, ϕ) with $P_C := L_C \cap S_0 \subset \nu(C)$ and $P_{C'} := L_{C'} \cap S_0 \subset \nu(C')$. If L_C and $L_{C'}$ are braid isotopic then $c(\phi_{L_C}, C) = c(\phi_{L_{C'}}, C)$.

Proof. By cutting $M_{(S, \phi)}$ along the page S_0 we get n -braids β_{L_C} and $\beta_{L_{C'}} \subset S \times [0, 1]$. Since L_C and $L_{C'}$ are braid isotopic we have

$$(2.3) \quad \beta_{L_{C'}} = \gamma^{-1} \beta_{L_C} \gamma^\phi \text{ (read from the right to left)}$$

for some braid $\gamma : \bigsqcup_{i=1}^n [0, 1] \rightarrow S \times [0, 1]$ connecting $P_{C'} \times \{0\}$ and $P_C \times \{1\}$, γ^{-1} is the braid γ with the reversed orientation, $\gamma^\phi : \bigsqcup_{i=1}^n [0, 1] \rightarrow S \times [0, 1]$ is a braid given by $\gamma^\phi(t) = \phi(\gamma(t))$, and the product in (2.3) is the concatenation of braids.

We regard the n -braid γ as an isotopy $\{\gamma_t : \{1, \dots, n\} \rightarrow S \mid t \in [0, 1]\}$ of ordered n points. We extend γ_t to an isotopy $\hat{\gamma}_t : S \rightarrow S$ of the surface and obtain a homeomorphism $\Gamma := \hat{\gamma}_1 : (S, P_{C'}) \rightarrow (S, P_C)$. The homeomorphism Γ is unique up to isotopy, and it gives rise to an isomorphism $\gamma_* : \mathcal{MCG}(S; P_{C'}) \rightarrow \mathcal{MCG}(S; P_C)$. By (2.3) the distinguished monodromies satisfy

$$(2.4) \quad \begin{aligned} \phi_{L_{C'}} &= i(\beta_{L_{C'}}) \circ j_*(\phi) \\ &= i(\gamma^{-1} \beta_{L_C} \gamma^\phi) \circ j_*(\phi) \\ &= \Gamma^{-1} \circ i(\beta_{L_C}) \circ (j_*(\phi) \circ \Gamma \circ j_*(\phi^{-1})) \circ j_*(\phi) \\ &= \Gamma^{-1} \circ \phi_{L_C} \circ \Gamma. \\ &= (\gamma_*)^{-1}(\phi_{L_C}) \end{aligned}$$

The FDTC map is natural with respect to move of puncture points, namely, the diagram

$$\begin{array}{ccc} \mathcal{MCG}(S; P_{C'}) & \xrightarrow{\gamma_*} & \mathcal{MCG}(S; P_C) \\ & \searrow c(-, C) \quad \swarrow c(-, C) & \\ & \mathbb{Q} & \end{array}$$

commutes. When $P_{C'} = P_C$ (i.e., $C = C'$) the isomorphism γ_* is nothing but an inner automorphism of $\mathcal{MCG}(S; P_C)$ and the commutativity means invariance of the FDTC under conjugation. Hence we have $c(\phi_{L_C}, C) = c(\gamma_*(\phi_{L_{C'}}), C) = c(\phi_{L_{C'}}, C)$. \square

If a braid L is empty we set $P = \emptyset$ and define the distinguished monodromy $\phi_L := \phi$. Hence the FDTC of the empty closed braid is equal to the FDTC of the monodromy of the open book.

2.2. Branched coverings and the FDTC. Given a contact 3-manifold (M, ξ) with a transverse link $L \subset (M, \xi)$ and a covering $\pi : \widetilde{M} \rightarrow M$ branched along L , there exists a contact structure $\widetilde{\xi}$ on \widetilde{M} unique up to isotopy such that $\pi_*(\widetilde{\xi})$ is isotopic to ξ through contact structures. See [23, Section 2] for a construction of $\widetilde{\xi}$ and its uniqueness. We call the contact 3-manifold $(\widetilde{M}, \widetilde{\xi})$ the *contact branched covering of (M, ξ) branched along L* .

Definition 2.4 (Branched coverings of open books). Let (S, ϕ) be an open book supporting the contact manifold $(M, \xi) := (M_{(S, \phi)}, \xi_{(S, \phi)})$. Let L be a closed n -braid with respect to (S, ϕ) and $P := L \cap S_0$. Let $\pi : \widetilde{S} \rightarrow S$ be a branched covering of S branched at the n -points $P \subset S$. Suppose that \widetilde{S} is connected. Put $\widetilde{P} := \pi^{-1}(P)$. We say that an open book $(\widetilde{S}, \widetilde{\phi})$ is a *branched covering of (S, ϕ) along the closed braid L* if there exists $\psi \in \mathcal{MCG}(\widetilde{S}; \widetilde{P})$ such that $f(\psi) = \widetilde{\phi}$ (the map f is the forgetful map in (2.1)) and

$$\pi \circ \psi = \phi_L \circ \pi$$

where ϕ_L is the distinguished monodromy introduced in Definition 2.1.

$$\begin{array}{ccc} (\widetilde{S}, \widetilde{P}) & \xrightarrow{\psi} & (\widetilde{S}, \widetilde{P}) \\ \pi \downarrow & & \downarrow \pi \\ (S, P) & \xrightarrow{\phi_L} & (S, P) \end{array}$$

The covering $\pi : \widetilde{S} \rightarrow S$ branched at P induces a covering $\pi : M_{(\widetilde{S}, \widetilde{\phi})} \rightarrow M_{(S, \phi)}$ branched along L . The contact 3-manifold $(M_{(\widetilde{S}, \widetilde{\phi})}, \xi_{(\widetilde{S}, \widetilde{\phi})})$ supported by $(\widetilde{S}, \widetilde{\phi})$ gives a contact branched covering of (M, ξ) branched along L . The preimage $\widetilde{L} := \pi^{-1}(L)$ is a closed braid with respect to the open book $(\widetilde{S}, \widetilde{\phi})$ and $\widetilde{\phi}_{\widetilde{L}} = \psi \in \mathcal{MCG}(\widetilde{S}; \widetilde{P})$.

The following proposition follows from the definition of the FDTC [15].

Proposition 2.5. *For a boundary component C of S let \widetilde{C} be a connected component of the preimage $\pi^{-1}(C)$. Let $d(\pi, \widetilde{C})$ denote the degree of the covering $\pi|_{\widetilde{C}} : \widetilde{C} \rightarrow C$. Assume that $\chi(\widetilde{S}) < 0$. Then $c(\widetilde{\phi}, \widetilde{L}, \widetilde{C}) = c(\widetilde{\phi}, \widetilde{C})$ and*

$$c(\widetilde{\phi}, \widetilde{L}, \widetilde{C}) \cdot d(\pi, \widetilde{C}) = c(\widetilde{\phi}, \widetilde{C}) \cdot d(\pi, \widetilde{C}) = c(\phi, L, C).$$

Remark 2.6. If $\chi(\widetilde{S}) = 0$ (i.e. \widetilde{S} is an annulus) the above formula does not hold. Consider a double branched covering $\pi : A \rightarrow D^2$ branched at two points $P = \{p_1, p_2\} \subset D^2$ and the positive half-twist $\sigma \in \mathcal{MCG}(D^2; P) \simeq B_2$. The mapping class σ lifts to the positive Dehn twist $\tau \in \mathcal{MCG}(A)$ along the core of the annulus A . We have $c(\tau, \widetilde{C}) = 1$, $d(\pi, \widetilde{C}) = 1$ and $c(\sigma, C) = \frac{1}{2}$.

Corollary 2.7. *Let $(\tilde{S}, \tilde{\phi})$ be a branched open book covering of (S, ϕ) branched along L . If $\chi(\tilde{S}) < 0$ and $c(\phi, L, C) < 0$ for some boundary component C then $(\tilde{S}, \tilde{\phi})$ supports an overtwisted contact structure.*

Proof. By Proposition 2.5 we have $c(\tilde{\phi}, \tilde{C}) < 0$ for a connected component \tilde{C} of the preimage of C . This means that $\tilde{\phi}$ is not right-veering. Hence Theorem 1.1 implies that $(\tilde{S}, \tilde{\phi})$ supports an overtwisted contact structure. \square

In general taking a branched cover does not preserve the geometric structure. For example, a branched cover of S^3 along a hyperbolic link is not necessarily hyperbolic. In the following corollary we give a sufficient condition on the FDTC that the geometric structure to be preserved under taking a branched cover.

Corollary 2.8. *Let $(\tilde{S}, \tilde{\phi})$ be a branched covering of (S, ϕ) branched along a closed braid L with $\chi(\tilde{S}) < 0$. Assume that $|c(\phi, L, C)| > 4d(\pi, \tilde{C})$ for every boundary component $C \subset \partial S$ and connected component $\tilde{C} \subset \pi^{-1}(C)$. If $M_{(S, \phi)} \setminus L$ is Seifert-fibered (resp. toroidal, hyperbolic) then $M_{(\tilde{S}, \tilde{\phi})}$ is Seifert-fibered (resp. toroidal, hyperbolic).*

Proof. We have $|c(\phi, L, C)| \geq |c(\phi, L, C)/d(\pi, \tilde{C})| > 4$. [19, Theorem 8.4] and $|c(\phi, L, C)| > 4$ yield that $M_{(S, \phi)} \setminus L$ is Seifert-fibered (resp. toroidal, hyperbolic) if and only if ϕ_L is periodic (resp. reducible, pseudo-Anosov). Since $\tilde{\phi}$ is a lift of ϕ_L and $\chi(\tilde{S}) < 0$ the map $\tilde{\phi}$ is periodic (resp. reducible, pseudo-Anosov).

By Proposition 2.5 $|c(\tilde{\phi}, \tilde{C})| = |c(\tilde{\phi}, \tilde{L}, \tilde{C})| = |c(\phi, L, C)/d(\pi, \tilde{C})| > 4$. With [19, Theorem 8.3] we conclude that $M_{(\tilde{S}, \tilde{\phi})}$ is Seifert-fibered (resp. toroidal, hyperbolic). \square

3. QUASI-RIGHT-VEERING MAPS

We use the notations of the previous section. Thus, when we discuss a closed braid L and its distinguished monodromy ϕ_L there exists a particular boundary component C of S such that L has $L \cap S_0 \subset \nu(C)$. The puncture set P is given by $P = L \cap S_0$.

For each boundary component C of S , we choose a base point $*_C \in C$. Let $\mathcal{A}_C(S; P)$ be the set of isotopy classes of properly embedded arcs $\gamma : [0, 1] \rightarrow S \setminus P$ satisfying $\gamma(0) = *_C$. We do not allow $\gamma \in \mathcal{A}_C(S; P)$ to have $\gamma(1) \in P$ but we allow $\gamma(1) \in (C \setminus \{*_C\})$. Abusing the notation, an element $\gamma \in \mathcal{A}_C(S; P)$ often means an actual arc $[0, 1] \rightarrow S$ representing γ and we may call an element of $\mathcal{A}_C(S; P)$ simply an *arc*.

We say that two arcs α and β intersect *efficiently* if they attain the minimal geometric intersection number among all the arcs isotopic to them.

Definition 3.1 (Right-veering total ordering \prec_{right}). Let $\alpha, \beta \in \mathcal{A}_C(S; P)$ be arcs intersecting efficiently. We denote $\alpha \prec_{\text{right}} \beta$ and say that β lies on the *right side* of α if the arc β lies on the right side of α in a small neighborhood of the base point $*_C$.

In [15] and [19], where the set P is empty, the symbol “ $>$ ” is used in the place of “ \prec_{right} ”.

The order “ \prec_{right} ” is a total ordering. For any family of arcs $\{\alpha_i\} \subset \mathcal{A}_C(S; P)$ we can always put them in a position simultaneously so that α_i and α_j intersect efficiently for any pairs (i, j) . This can be done, for example, by choosing a hyperbolic metric on $S \setminus P$ and realizing the arcs as geodesics.

Naturally extending the notion of right-veering in [15] we define the following, cf. [3, p.949]:

Definition 3.2 (Right-veering).

- We say that $\psi \in MCG(S; P)$ is *right-veering* with respect to the boundary component C if $\alpha \prec_{\text{right}} \psi(\alpha)$ or $\alpha = \psi(\alpha)$ for all $\alpha \in \mathcal{A}_C(S; P)$. Since \prec_{right} is a total ordering on the set $\mathcal{A}_C(S; P)$, $\psi \in MCG(S; P)$ being right-veering is equivalent to saying that any arc $\alpha \in \mathcal{A}_C(S; P)$ has $\psi(\alpha) \not\prec_{\text{right}} \alpha$.
- We also say that a closed braid L in an open book (S, ϕ) (with $P := L \cap S_0 \subset \nu(C)$) is *right-veering* with respect to C if $\phi_L \in MCG(S; P)$ is right-veering with respect to C . As we have seen in (2.4) in the proof of Proposition 2.3 the right-veering property of ϕ_L is independent of the choice of $P = L \cap S_0$.

Remark 3.3. In [2, 25], a slightly different definition of “right-veering” is used. See Section 7 for the relationship between these two superficially different notions of right-veering.

We will define another ordering “ \ll_{right} ” which plays a central role in this paper.

Definition 3.4 (Strongly right-veering partial ordering \ll_{right}). For two arcs $\alpha, \beta \in \mathcal{A}_C(S; P)$, we define $\alpha \ll_{\text{right}} \beta$ if there exists a sequence of arcs $\alpha_0, \dots, \alpha_k \in \mathcal{A}_C(S; P)$ such that

$$(3.1) \quad \alpha = \alpha_0 \prec_{\text{right}} \alpha_1 \prec_{\text{right}} \dots \prec_{\text{right}} \alpha_k = \beta \text{ in } \mathcal{A}_C(S; P), \text{ and}$$

$$(3.2) \quad \text{Int}(\alpha_i) \cap \text{Int}(\alpha_{i+1}) = \emptyset \text{ for all } i = 0, \dots, k-1.$$

By the definition it is easy to see that \ll_{right} is a partial ordering, i.e., $\alpha \ll_{\text{right}} \beta$ and $\beta \ll_{\text{right}} \gamma$ imply $\alpha \ll_{\text{right}} \gamma$. If the puncture set P is empty, by [15, Lemma 5.2] the ordering \ll_{right} coincides with \prec_{right} . On the other hand, when P is non-empty \ll_{right} is *not* a total ordering and there is a difference between \prec_{right} and \ll_{right} . To see the difference we use the following notion.

Definition 3.5 (Boundary right P -bigon). Let $\alpha, \beta \in \mathcal{A}_C(S; P)$ with $\alpha \prec_{\text{right}} \beta$. Assume that there exist subarcs $\delta_\alpha \subset \alpha$ and $\delta_\beta \subset \beta$ such that

- $*_C \in \delta_\alpha \cap \delta_\beta$
- $\delta_\alpha \cup \delta_\beta$ bounds a (possibly immersed) bigon $D(\subset S)$ which lies on the right side of α (i.e., the orientation of δ_α , as a subarc of α , disagrees with the orientation of ∂D) and
- D contains some of the marked points of P .

We call such a bigon D a *boundary right P -bigon from α to β* .

A boundary right P -bigon gives an obstruction for $\alpha \ll_{\text{right}} \beta$:

Proposition 3.6. *Let $\alpha, \beta \in \mathcal{A}_C(S; P)$ be arcs with $\alpha \prec_{\text{right}} \beta$. If there is a boundary right P -bigon from α to β then $\alpha \not\ll_{\text{right}} \beta$.*

Proof. If there is a boundary right P -bigon D from α to β then every arc $\gamma \in \mathcal{A}_C(S; P)$ that satisfies $\alpha \prec_{\text{right}} \gamma \prec_{\text{right}} \beta$ must intersect D and yields either a boundary right P -bigon from α to γ or from γ to β (see Figure 1 (a)). Thus for any sequence of arcs $\alpha = \gamma_0 \prec_{\text{right}} \gamma_1 \prec_{\text{right}} \dots \prec_{\text{right}} \gamma_n = \beta$ for some i there exists a boundary right P -bigon from γ_i to γ_{i+1} , which means $\text{Int}(\gamma_i)$ and $\text{Int}(\gamma_{i+1})$ cannot be disjoint. \square

We now clearly see the difference between \prec_{right} and \ll_{right} . Namely $\alpha \prec_{\text{right}} \beta$ and $\beta \ll_{\text{right}} \gamma$ (and also $\alpha \ll_{\text{right}} \beta$ and $\beta \prec_{\text{right}} \gamma$) do *not* imply $\alpha \ll_{\text{right}} \gamma$. For example, the arcs depicted in Figure 1 (b) satisfy $\alpha \prec_{\text{right}} \beta$ and $\beta \ll_{\text{right}} \gamma$ but by Proposition 3.6 $\alpha \not\ll_{\text{right}} \gamma$.

We conjecture the converse of Proposition 3.6:

Conjecture 3.7. *$\alpha \ll_{\text{right}} \beta$ if and only if $\alpha \prec_{\text{right}} \beta$ and there exist no boundary right P -bigons from α to β .*

The next lemma easily follows from the definition of \ll_{right} .

Lemma 3.8. *Let $f : \mathcal{A}_C(S; P) \rightarrow \mathcal{A}_C(S)$ be the forgetful map induced by the obvious inclusion $S \setminus P \hookrightarrow S$. If $\alpha \ll_{\text{right}} \beta$ in $\mathcal{A}_C(S; P)$ then we have $f(\alpha) \prec_{\text{right}} f(\beta)$ in $\mathcal{A}_C(S)$.*

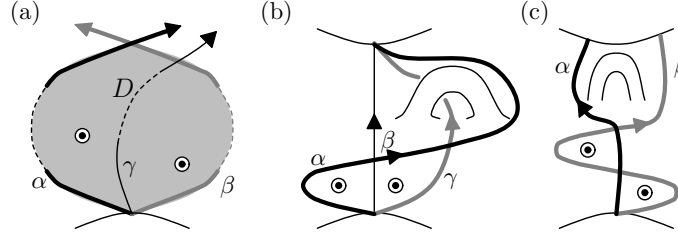


FIGURE 1. (a) The arc γ with $\alpha \prec_{\text{right}} \gamma \prec_{\text{right}} \beta$ cuts the boundary right P -bigon D , yielding a boundary right P -bigon from α to γ , or from β to γ .
 (b) $\alpha \prec_{\text{right}} \beta$ and $\beta \ll_{\text{right}} \gamma$, but $\alpha \not\prec_{\text{right}} \gamma$.
 (c) $f(\alpha) \prec_{\text{right}} f(\beta)$ and $\alpha \prec_{\text{right}} \beta$, but $\alpha \not\prec_{\text{right}} \beta$.

Proof. Since $\alpha \ll_{\text{right}} \beta$, there is a sequence of arcs $\alpha = \gamma_0 \prec_{\text{right}} \gamma_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \gamma_n = \beta$ with $\text{Int}(\gamma_i) \cap \text{Int}(\gamma_{i+1}) = \emptyset$ for all i . This implies that γ_i and γ_{i+1} do not cobound marked bigons and $f(\gamma_i) \cap f(\gamma_{i+1}) = \emptyset$. Hence we conclude $f(\alpha) = f(\gamma_0) \prec_{\text{right}} f(\gamma_1) \prec_{\text{right}} \cdots \prec_{\text{right}} f(\gamma_n) = f(\beta)$ in $\mathcal{A}_C(S)$ that is $f(\alpha) \prec_{\text{right}} f(\beta)$ in $\mathcal{A}_C(S)$. \square

Remark. The converse of the lemma does not hold in general, even if we assume $\alpha \prec_{\text{right}} \beta$. See Figure 1 (c).

The next proposition gives a sufficient condition for $\alpha \ll_{\text{right}} \beta$.

Proposition 3.9. *Let $\alpha, \beta \in \mathcal{A}_C(S; P)$ be arcs with $\alpha \prec_{\text{right}} \beta$. If α and β do not cobound bigons with marked points (namely, if α and β , viewed as arcs in the non-punctured surface S , intersect efficiently), then $\alpha \ll_{\text{right}} \beta$.*

Proof. If α and β do not cobound bigons with marked points then following the proof of [15, Lemma 5.2] one can construct an arc $\gamma \in \mathcal{A}_C(S; P)$ such that $\alpha \prec_{\text{right}} \gamma \prec_{\text{right}} \beta$ with $\#(\alpha, \gamma) < \#(\alpha, \beta)$ and $\#(\gamma, \beta) < \#(\alpha, \beta)$. Here $\#(-, -)$ denotes the geometric intersection number of the interiors of the two arcs. Moreover, the construction of γ shows that α and γ (γ and β) do not cobound bigons with marked points. Thus iterating this interpolation process, we get a sequence of arcs with (3.1) and (3.2). \square

Now we introduce quasi right-veering mapping classes and quasi right-veering closed braids.

Definition 3.10. (Quasi right-veering)

- We say that $\psi \in \mathcal{MCG}(S; P)$ is *quasi right-veering* with respect to the boundary component C if any arc $\alpha \in \mathcal{A}_C(S; P)$ satisfies $\psi(\alpha) \not\prec_{\text{right}} \alpha$. (Since “ \ll_{right} ” is not a total ordering, $\psi(\alpha) \not\prec_{\text{right}} \alpha$ is not equivalent to $\alpha \ll_{\text{right}} \psi(\alpha)$ or $\alpha = \psi(\alpha)$.)
- We say that a closed braid L in an open book (S, ϕ) is *quasi right-veering* with respect to a boundary component C if its distinguished monodromy $\phi_L \in \mathcal{MCG}(S; P)$ is quasi right-veering with respect to C .
- We say that L is *quasi right-veering* if L is quasi right-veering with respect to all the boundary components of S .

If L is empty then by definition $\phi_L = \phi$. That is, the empty closed braid is quasi right-veering if and only if the monodromy ϕ is right-veering.

We note that the definitions of “right-veering” and “quasi right-veering” are independent of a choice of the distinguished point $*_C$.

Proposition 3.11. *A mapping class $\psi \in \mathcal{MCG}(S; P)$ is quasi right-veering if ψ is right-veering. More generally, $\psi \in \mathcal{MCG}(S; P)$ is quasi right-veering if $f(\psi) \in \mathcal{MCG}(S)$ is right-veering, where $f : \mathcal{MCG}(S; P) \rightarrow \mathcal{MCG}(S)$ is the forgetful map in the generalized Birman exact sequence (2.1).*

As a consequence, every closed braid L in an open book (S, ϕ) is quasi-right-veering if $\phi \in \mathcal{MCG}(S)$ is right-veering. In particular, every closed braid in the open book (D^2, id) is quasi-right-veering.

Proof. Assume that $\psi \in \mathcal{MCG}(S; P)$ is not quasi right-veering with respect to some boundary component C of S . Then there exists an arc $\alpha \in \mathcal{A}_C(S; P)$ such that $\psi(\alpha) \ll_{\text{right}} \alpha$. By Lemma 3.8 we get $f(\psi)(f(\alpha)) = f(\psi(\alpha)) \prec_{\text{right}} f(\alpha)$ in $\mathcal{A}_C(S; P)$, that is, $f(\psi) \in \mathcal{MCG}(S)$ is not right-veering. \square

It is proved in [15, Section 3] that the right-veeringness of $\phi \in \mathcal{MCG}(S)$ is almost equivalent to positivity of its FDTC. We say “almost” because the statement is slightly complicated when the FDTC = 0 for non-pseudo Anosov case. If $\phi \in \mathcal{MCG}(S)$ is pseudo Anosov, ϕ is right-veering with respect to a boundary component C if and only if $c(\phi, C) > 0$. We remark that parallel statements on positivity and right-veering-ness hold for elements $\psi \in \mathcal{MCG}(S, P)$. Namely if ψ is right-veering then $c(\psi, C) \geq 0$. Moreover, if ψ is pseudo Anosov then ψ is right-veering with respect C if and only if $c(\psi, C) > 0$.

The next proposition demonstrates the significant difference between quasi right-veering and right-veering provided that P is non-empty. In particular, quasi right-veering property is much less related to positivity of the FDTC.

Proposition 3.12. *Let (S, ϕ) be an open book.*

- (1) *For a boundary component C of S and integers $N < 0$ and $n > 1$, there exists a closed n -braid L with respect to (S, ϕ) that is quasi right-veering with respect to C but not right-veering with $c(L, \phi, C) \leq N$.*
- (2) *For any negative integer N there exists a closed braid L with respect to (S, ϕ) which is quasi right-veering and $c(L, \phi, C) \leq N$ for every boundary component C of S .*

Proof. Take a collar neighborhood $\nu(C)$ of a boundary component C of S so that $\phi = id$ on $\nu(C)$. We identify $\nu(C)$ with the annulus $A = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ so that the boundary component C is identified with $\{z \in \mathbb{C} \mid |z| = 1\}$. We put

$$P = \left\{ p_1, \dots, p_n \in \mathbb{C} \mid p_i = 1 + \frac{i}{n+1} \ (i = 1, \dots, n) \right\} \subset A \cong \nu(C) \subset S.$$

For $k \in \mathbb{N}$ let $\beta_{C,k}$ be the n -braid in $S \times [0, 1]$ whose i -th strand $\gamma_{k,i} : [0, 1] \rightarrow A \times [0, 1] \subset S \times [0, 1]$ is given by (see Figure 2-(1))

$$\gamma_{k,i}(t) = \begin{cases} ((1 + \frac{1}{n+1}) \exp(2\pi\sqrt{-1}kt), t) & (i = 1) \\ ((1 + \frac{2}{n+1}) \exp(-2\pi\sqrt{-1}kt), t) & (i = 2) \\ (1 + \frac{i}{n+1}, t) & (i = 3, \dots, n). \end{cases}$$

Thus, the 1st strand of $\beta_{C,k}$ winds k times around C counterclockwise and the 2nd strand winds k times clockwise. Let $L_{C,k}$ be the closed braid in the open book (S, ϕ) obtained by closing the braid $\beta_{C,k}$.

With the map $i : B_n(S) \rightarrow \mathcal{MCG}(S; P)$ in (2.1) we have $i(\beta_{C,1}) = (T_C)^{-1}(T_{C'})^2(T_{C''})^{-1}$, where $T_C, T_{C'}$ and $T_{C''}$ are the right-handed Dehn twists along the curves $C, C' = \{z \in A \mid |z| = \frac{3}{2n+2}\}$ and $C'' = \{z \in A \mid |z| = \frac{5}{2n+2}\}$. The distinguished monodromy of the closed braid $L := L_{C,k}$ is

$$\phi_L = \phi_{L_{C,k}} = i(\beta_{C,k})j_*(\phi) = (T_C)^{-k} (T_{C'})^{2k} (T_{C''})^{-k} j_*(\phi) \in \mathcal{MCG}(S; P).$$

Since $j_*(\phi) = id$ on $\nu(C)$ we have $c(L, \phi, C) = -k$.

For any $\alpha \in \mathcal{A}_C(S; P)$ the factor $(T_C)^{-k}(T_{C'})^{2k}(T_{C''})^{-k}$ of ϕ_L forces to form a boundary right P -bigon from $\phi_L(\alpha)$ to α . See Figure 2-(2). Thus $\phi_L(\gamma) \not\ll_{\text{right}} \gamma$ for any $\gamma \in \mathcal{A}_C(S; P)$, which means $L_{C,k}$ is quasi right-veering. This proves (1).

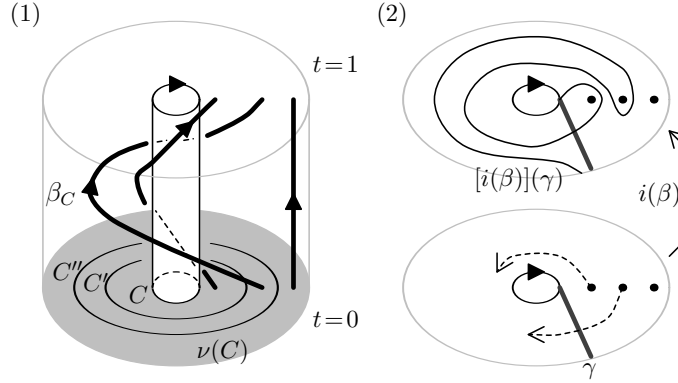


FIGURE 2. (1) The braid $L_{C,1}$ is not right-veering but quasi right-veering.
 (2) The map $(T_C)^{-1}(T_{C'})^2(T_{C''})^{-1}$ forces to form a boundary right P -bigon.

Next we prove (2). Let C_1, \dots, C_d be the set of boundary components of S . For each component C_i we take a closed braid $L_{C_i,k}$ given in the proof of (1), and let $L = \bigsqcup_{i=1}^d L_{C_i,k}$ be the disjoint union of $L_{C_i,k}$. By (1) and Proposition 2.3 L is quasi right-veering and $c(L, \phi, C_i) \leq -k$ for all $i = 1, \dots, d$. \square

Corollary 3.13. *The set of quasi right-veering mapping classes in $\mathcal{MCG}(S; P)$ does not form a monoid.*

Proof. We use the same notations in Proposition 3.12. Let $\chi = (T_{C'})^{-1}i(\beta_{C,1})^{-1} = T_C T_{C'}^{-3} T_{C''}$ and $\psi = i(\beta_{C,1})$. Both χ and ψ are quasi right-veering but $\chi\psi = (T_{C'})^{-1}$ is not quasi right-veering. \square

Proposition 6.1 of [15] implies that every contact 3-manifold admits an open book decomposition (S, ϕ) with right-veering monodromy. Thus in light of Proposition 3.11 every transverse link in (M, ξ) admits a quasi right-veering closed braid representative with respect to some open book decomposition of (M, ξ) . The next proposition shows more is true.

Proposition 3.14. *Every closed braid L in an open book (S, ϕ) can be made right-veering after a sequence of positive stabilizations.*

When $(S, \phi) = (D^2, id)$ the same statement is proved in [25, Proposition 3.1].

Proof. As usual, we put L so that $P = L \cap S_0$ is contained in a collar neighborhood of some boundary component, say C_0 . Let C be a boundary component of S which may or may not be the same as C_0 . Let $\nu'(C) \subset (\nu(C) \setminus P)$ be a sub-collar neighborhood of C such that $\phi_L = id$ on $\nu'(C)$. Choose points q and q' in $\nu'(C)$. Let $\gamma_1 \subset S$ be an arc that connects one of the puncture points in P and the point q . Let $\gamma_2 \subset \nu'(C)$ be an arc that connects q and $q' \in \nu'(C)$ and satisfying the following (see Figure 3)

- (1) The interiors of γ_1 and γ_2 intersect exactly at one point, say $r \in \nu'(C)$.
- (2) Let $\gamma'_1 \subset \gamma_1$ and $\gamma'_2 \subset \gamma_2$ be the sub-arcs connecting r and q . Then the simple closed curve $\gamma'_1 \cup \gamma'_2$ is homotopic to C .

Let L' be a closed braid obtained from L by positive stabilizations first along γ_1 and then γ_2 . The distinguished monodromy of L' is $\phi_{L'} = H_{\gamma_2} \circ H_{\gamma_1} \circ \phi_L$, where H_{γ_i} is the positive half twist along the arc γ_i . Since $\phi_L = id$ on $\nu'(C)$ and every essential arc starting from C intersects either γ_1 or γ_2 , the monodromy $\phi_{L'}$ is right-veering with respect to C .

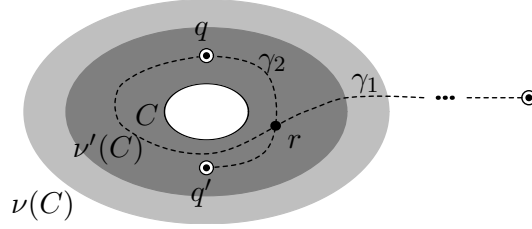


FIGURE 3. Twice stabilizations about C makes a closed braid right-veering with respect to C .

Although $\phi_{L'}$ can be defined as an element of $\mathcal{MCG}(S; P)$, strictly speaking if $C_0 \neq C$, $\phi_{L'}$ does not satisfy Definition 2.1 because $L' \cap S_0 \not\subseteq \nu(C)$. In that case we move L' by braid isotopy so that the resulting braid, L'' , has all the punctures in $\nu(C)$. The argument in the proof of Proposition 2.3 shows that this does not change the right-veeringness. Namely, $\phi_{L''}$ is right-veering with respect to C if and only if $\phi_{L'}$ is right-veering with respect to C .

Applying this operation for every boundary component we get a right-veering closed braid that is transversely isotopic to the original braid L . \square

4. PROOF OF THEOREM 1.2

We now prove Theorem 1.2, which we restate here for the sake of readability.

Theorem 1.2 *A transverse link K in a contact 3-manifold (M, ξ) is non-loose if and only if every closed braid representative of K with respect to an arbitrary open book decomposition that supports (M, ξ) is quasi right-veering.*

Our proof of Theorem 1.2 is a generalization of the proof of [18, Theorem 2.4]. We use open book foliations that are defined and studied in [17, 19, 20]. Basic machinery of open book foliations can be found for example in [17, Section 2].

Proof of Theorem 1.2. (\Rightarrow) First we show that non-quasi-right-veering braid is loose. Assume that a transverse link K can be represented by a non-quasi-right-veering closed L with respect to an open book (S, ϕ) . That is, there exist a boundary component $C \subset \partial S$ and an arc $\alpha \in \mathcal{A}_C(S; P)$ such that there is a sequence of arcs $\phi_L(\alpha) = \alpha_0 \prec_{\text{right}} \alpha_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \alpha_k = \alpha$ with $\text{Int}(\alpha_i) \cap \text{Int}(\alpha_{i+1}) = \emptyset$ for all $i = 0, \dots, k-1$.

We explicitly construct a transverse overtwisted disk D_{trans} in $M \setminus L$ by giving its movie presentation. A similar construction can be found in [18]. Here, a *transverse overtwisted disk* (see [17, Definition 4.1] for the precise definition) is a disk admitting a certain types of open book foliation and is bounded by a transverse push-off of a usual overtwisted disk.

For $i = 0, \dots, k$ denote the endpoint $\alpha_i(1) \in \partial S$ of the arc α_i by w_i . Slightly moving w_i along ∂S , if necessary, we may assume that all the points w_0, \dots, w_{k-1} are distinct and satisfying $\text{Int}(\alpha_i) \cap \text{Int}(\alpha_{i+1}) = \emptyset$ and $w_0 = w_k$. Fix a sufficiently small $\varepsilon > 0$.

The open book foliation of D_{trans} contains one negative elliptic point at $*_C$ and k positive elliptic points at w_0, \dots, w_{k-1} .

The movie presentation of D_{trans} on the page S_0 consists of $(k-1)$ a-arcs emanating from w_1, \dots, w_{k-1} and a b-arc that is a copy of α_0 joining w_0 and $*_C$. For $t \in [0, \frac{1}{k+1})$ the movie presentation on the page S_t is the same as S_0 .

The movie presentation on $S_{\frac{1}{k+1}}$ contains one hyperbolic point, h_1 , whose describing arc is a parallel copy of α_1 in $S_{\frac{1}{k+1}-\varepsilon}$. (In Figure 4 the black dashed arc is the describing arc and the

gray dashed arc is α_0 . The gray dashed arrows indicate the normal vectors to D_{trans} .) Since $\text{Int}(\alpha_0) \cap \text{Int}(\alpha_1) = \emptyset$ the interior of the describing arc is disjoint from all of the a-arcs and the b-arc in the page $S_{\frac{1}{k+1}-\varepsilon}$. Since $\alpha_0 \prec_{\text{right}} \alpha_1$ the normals of D_{trans} point out of the describing arc, thus the sign of the hyperbolic point h_1 is positive (see Observation 2.5 of [21]). The movie presentation on the page $S_{\frac{1}{k+1}+\varepsilon}$ consists of one b-arc which is a copy of α_1 connecting w_1 and $*_C$ and $(k-1)$ a-arcs emanating from w_0, w_2, \dots, w_{k-1} .

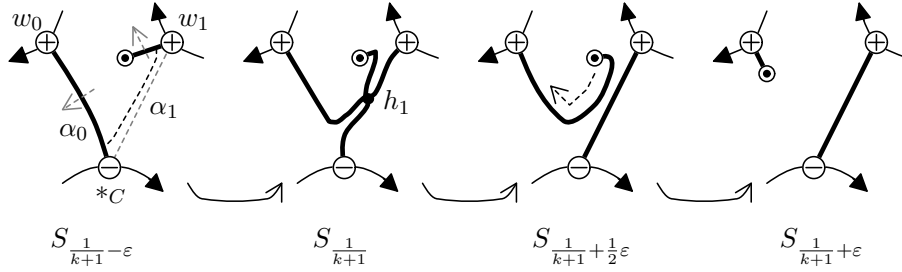


FIGURE 4. Movie in $[\frac{1}{k+1} - \varepsilon, \frac{1}{k+1} + \varepsilon]$: the b-arc is changed from α_0 to α_1 .

We inductively apply the same procedure: On the page $S_{\frac{j}{k+1}}$ ($j > 1$) we put a positive hyperbolic point h_j whose describing arc is a parallel copy of α_j . As a consequence the page $S_{\frac{j}{k+1}+\varepsilon}$ has one b-arc which is a copy of α_j connecting w_j and $*_C$ and $(k-1)$ a-arcs emanating from w_i ($i = 0, \dots, j, \dots, k-1$).

On the page S_1 the movie presentation consists of one b-arc which is a copy of $\alpha_k = \alpha$ and $(k-1)$ a-arcs emanating from w_0, \dots, w_{k-1} . Since $\phi_L(\alpha) = \alpha_0$ the slices $D_{\text{trans}} \cap S_1$ and $D_{\text{trans}} \cap S_0$ of D_{trans} can be identified under the distinguished monodromy ϕ_L . In other words the movie presentation gives rise to an embedded surface in $M \setminus L$. The construction tells us that the surface is topologically a disk and moreover is a transverse overtwisted disk (see [18]).

(\Leftarrow) Assume that a transverse link $L \subset (M, \xi)$ is loose. By taking a neighborhood of an overtwisted disk $D \subset M \setminus L$, we may regard (M, ξ) as the connected sum $(M', \xi') \# (S^3, \xi'_{ot})$ such that $L \subset (M', \xi')$. Here ξ'_{ot} denotes some overtwisted contact structure on S^3 . By further taking connected sum decomposition of (S^3, ξ'_{ot}) , if necessary, we may regard (M, ξ) as $(N, \xi_N) \# (S^3, \xi_{ot})$ such that $L \subset (N, \xi_N)$, where (S^3, ξ_{ot}) denotes the overtwisted contact structure supported by the annulus open book (A, T_A^{-1}) with the left-handed Dehn twist.

Take an open book decomposition (S_N, ϕ_N) of (N, ξ_N) and a closed braid representative L_N of L . Then the original contact 3-manifold (M, ξ) is supported by the open book $(S, \phi) := (S_N, \phi_N) * (A, T_A^{-1})$ and L_N is in a braid position with respect to (S, ϕ) . For the co-core $\gamma \subset A$ of the attached 1-handle we have $\phi_{L_N}(\gamma) = \phi(\gamma) \ll_{\text{right}} \gamma$ hence L_N is not quasi right-veering. \square

The following statement is a weakened version of Theorem 1.2 but still gives a criterion of non-loose links.

Corollary 4.1. *A transverse link K in a contact 3-manifold (M, ξ) is non-loose if and only if for every closed braid representative b of K with respect to an arbitrary open book decomposition (S, ϕ) that supports (M, ξ) and for every properly embedded arc $\gamma \in S \setminus P$, at least one of the following holds:*

- (1) $\gamma = \phi_b(\gamma)$.
- (2) $\gamma \prec_{\text{right}} \phi_b(\gamma)$.
- (3) γ and $\phi_b(\gamma)$ cobound bigons that contain points of P .

Proof. (\Rightarrow) If there exists γ such that $\phi_b(\gamma) \prec_{\text{right}} \gamma$ and no marked bigons are cobounded by $\phi_b(\gamma)$ and γ then Proposition 3.9 shows that $\phi_b(\gamma) \ll_{\text{right}} \gamma$. Thus $\phi_b \in \mathcal{MCG}(S; P)$ is quasi-right-veering. Then Theorem 1.2 shows that K is non-loose.

(\Leftarrow) This implication holds by exactly the same proof of (\Leftarrow part of) Theorem 1.2. \square

5. DEPTH OF TRANSVERSE LINKS

Theorem 1.2 can be used to study the *depth* that measures non-looseness of transverse links and is introduced by Baker and Onaran in [1].

Let F be an oriented surface a 3-manifold and K be an oriented link that transversely intersects F . We denote the number of intersection points of K and F by $\#(K \cap F)$. We also denote the number of positive and negative intersection points of K and D by $\#^+(K \cap F)$ and $\#^-(K \cap F)$, respectively. They are not necessarily realizing the geometric intersection number and we have $\#(K \cap F) = \#^+(K \cap F) + \#^-(K \cap F)$.

The depth $d(K)$ of a transverse link K in (M, ξ) is defined by

$$d(K) = \min\{\#(K \cap D) \mid D \text{ is an overtwisted disk in } (M, \xi)\},$$

Thus K is loose if and only if $d(K) = 0$.

First we show that the depth of K is equal to the minimal number of the negative intersection points of K with a *transverse* overtwisted disk ([17, Definition 4.1]). The same result is proved in [22] for the case when K is the binding of an open book.

Theorem 5.1. *Let (S, ϕ) be an open book supporting a contact 3-manifold (M, ξ) . Let K be a transverse link in (M, ξ) . We have:*

$$(5.1) \quad d(K) = \min \left\{ \#^-(K' \cap D) \mid \begin{array}{l} K' \text{ is a link transversely isotopic to } K, \\ D \text{ is a transverse overtwisted disk in } (S, \phi). \end{array} \right\}$$

Proof. We denote by $d_{\text{trans}}(K)$ the quantity in the right hand side of (5.1). We first show that $d(K) \leq d_{\text{trans}}(K)$.

Let D_{trans} and K_0 be a transverse overtwisted disk and a transverse link which attains $d_{\text{trans}}(K)$. Therefore, $d_{\text{trans}}(K) = \#^-(K_0 \cap D_{\text{trans}})$. By the structural stability theorem [17, Theorem 2.21], we may assume that

- (a) The characteristic foliation $\mathcal{F}_\xi(D_{\text{trans}})$ and the open book foliation $\mathcal{F}_{ob}(D_{\text{trans}})$ are topologically conjugate.

Let $G_{++}(\mathcal{F}_\xi(D_{\text{trans}}))$ (resp. $G_{--}(\mathcal{F}_\xi(D_{\text{trans}}))$) be the Giroux graph consisting of the positive (resp. negative) elliptic points and the stable (resp. unstable) separatrices of positive (resp. negative) hyperbolic points (see [14, Page 646]). By the assumption (a), these graphs are identified with the corresponding graph G_{++} and G_{--} in the open book foliation (see [17, Definition 2.17] for the definitions).

Take small neighborhoods $N_+, N_- \subset D_{\text{trans}}$ of the graphs $G_{++}(\mathcal{F}_\xi(D_{\text{trans}}))$ and $G_{--}(\mathcal{F}_\xi(D_{\text{trans}}))$, respectively. By transverse isotopy of we move the intersection points $K_0 \cap D_{\text{trans}}$ without introducing new points so that:

- (b) The intersection $K_0 \cap D_{\text{trans}}$ is disjoint from the region $N_+ \cup N_-$.

We apply Giroux elimination lemma [13, Lemma 3.3] to remove all the positive elliptic and positive hyperbolic points of $\mathcal{F}_\xi(D_{\text{trans}})$. Call the resulting disk D' . By (a) and the definition of a transverse overtwisted disk, the characteristic foliation $\mathcal{F}_\xi(D')$ has a unique negative elliptic point enclosed by a circle leaf. We can find a usual overtwisted disc $D \subset D'$ (see Figure 5). Since the

Giroux elimination is supported on $N_+ \cup N_-$, the condition (b) implies that this process does not produce new intersections, i.e., $K_0 \cap D_{\text{trans}} = K_0 \cap D'$.

The proof of [1, Theorem 4.1.4] shows that every positive intersection of a Legendrian link and a (usual) overtwisted disk can be removed by a negative stabilization of the Legendrian link. Also the set of transverse links up to transverse isotopy is naturally identified, through the positive transverse push-off, with the set of Legendrian links up to Legendrian isotopy and negative stabilization [8, 10].

Therefore each positive intersection of K_0 and the overtwisted disk D can be removed by a suitable transverse isotopy. That is, there exists a link K_1 that is transversely isotopic to K_0 such that $\#(K_1 \cap D) = \#^-(K_1 \cap D) = \#^-(K_0 \cap D)$. We conclude

$$d(K) \leq \#(K_1 \cap D) = \#^-(K_0 \cap D) \leq \#^-(K_0 \cap D') = \#^-(K_0 \cap D_{\text{trans}}) = d_{\text{trans}}(K).$$

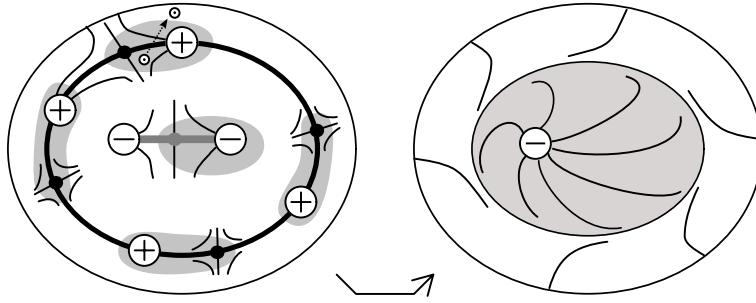


FIGURE 5. From a transverse overtwisted disk to a usual overtwisted disk. The graphs G_{++} and G_{--} are depicted by black and gray bold lines, respectively. The dots \odot represent the intersection points $K \cap D_{\text{trans}}$. They are moved away from the gray regions before applying the Giroux elimination lemma.

Next we show that $d(K) \geq d_{\text{trans}}(K)$. Let D be an overtwisted disk in (M, ξ) that intersects K at $d(K)$ points.

Take a slightly larger disc, D' , which contains D in its interior and is bounded by a positive transverse push-off of the Legendrian unknot ∂D so that $D' \cap K = D \cap K$.

Using transverse isotopy we make K disjoint from the binding of the open book. Following Pavalescu's proof of Alexander theorem [24, Theorem 3.2] one can find an isotopy of M preserving each page of the open book set-wise and taking the non-braided part of $\partial D \cup K$ (subsets which are not positively transverse to pages) into a neighborhood of the binding.

Inside the neighborhood of the binding we make $\partial D' \cup K$ braided with respect to the open book using [4]. We call the resulting link and disk K' and D'' , respectively. It is possible that new positive intersection points of D'' and K' may be created if a component of K is transversely isotopic to a binding component. However no new negative intersection points will be introduced. Hence $\#^-(K' \cap D'') = \#^-(K \cap D') \leq d(K)$.

Fixing $\partial D''$ and K' and following the proof of [19, Theorem 3.3] we perturb D'' so that the resulting disk, D''' , admits an essential open book foliation. This process can be done without introducing new intersection points with K' hence $\#^-(K' \cap D''') = \#^-(K' \cap D'')$.

Since the Bennequin-Eliashberg inequality does not hold

$$\text{sl}(\partial D''', [D''']) = \text{sl}(\partial D'', [D'']) = \text{sl}(\partial D', [D']) = \text{tb}(\partial D, [D]) - \text{rot}(\partial D, [D]) = 1 \not\leq -\chi(D''')$$

we can apply the proof of [17, Theorem 4.3] to D''' and obtain a transverse overtwisted disc, D_{trans} . By the nature of this construction we have

$$(5.2) \quad \begin{aligned} \#^-(K' \cap D_{\text{trans}}) &= \#^-(K' \cap D''') \\ \#^+(K' \cap D_{\text{trans}}) &\geq \#^+(K' \cap D''') \end{aligned}$$

where a strict inequality ' $>$ ' in (5.2) may hold only when a component of K' is transversely isotopic to a binding component. Summing up, we have

$$d_{\text{trans}}(K) \leq \#^-(K' \cap D_{\text{trans}}) = \#^-(K' \cap D''') = \#^-(K' \cap D'') = \#^-(K \cap D') \leq d(K).$$

□

The following theorem characterizes depth-one links containing the binding.

Theorem 5.2. *Let K be a transverse link in (M, ξ) . Let (S, ϕ) be an open book supporting (M, ξ) . Suppose that $K = B \cup L$ the union of the binding, B , of (S, ϕ) and a closed braid L with respect to (S, ϕ) . Then $d(K) = 1$ if and only if the braid L is non-quasi-right-veering.*

This theorem is a generalization of [22, Corollary 3.4], in which the closed braid L is empty. That is, for the binding B of an open book (S, ϕ) $d(B) = 1$ if and only if ϕ is not right-veering.

Proof. (\Leftarrow) Suppose that the braid L is non-quasi-right-veering. As in the proof of Theorem 1.2, we can construct a transverse overtwisted disk with only one negative elliptic point in the complement of L . By Theorem 5.1 we have $d(K) \leq 1$. Since the binding of any open book is non-loose [11] and K contains the binding B we have $d(K) \geq d(B) \geq 1$.

(\Rightarrow) Assume that $d(K) = 1$. Again by [11] there exists an overtwisted disk D such that $1 = \#(D, K) = \#(D, B)$ and $\#(D, L) = 0$. By the proof of Theorem 5.1 (the part showing $d(K) \geq d_{\text{trans}}(K)$) there exists a transverse overtwisted disk D_{trans} in the complement of L such that

$$\#^-(K, D_{\text{trans}}) = \#^-(B, D_{\text{trans}}) = 1.$$

Let $v \in B \cap D_{\text{trans}}$ denote the unique negative intersection point. That is, v is the unique negative elliptic point in the open book foliation $\mathcal{F}_{\text{ob}}(D_{\text{trans}})$ of D_{trans} . Assume that v lies on a boundary component C of S . For a regular page S_t of the open book let $b_t \in S_t$ be the unique b-arc in $\mathcal{F}_{\text{ob}}(D_{\text{trans}})$ that ends at v . We use v as the base point $*_C$ of C . Let $\pi : \overline{M \setminus (B \cup S_0)} \cong S \times [0, 1] \rightarrow S$ be the projection. We view the image $\pi(b_t)$ as an element of $\mathcal{A}_C(S; P)$ where $P = \pi(L \cap S_0)$ is a set of punctures given by the intersection of the braid L and the page S_0 . The equations (2.4) in the proof of Proposition 2.3 show that it is not necessary for the rest of the argument that P is included in a neighborhood of some boundary component.

Let S_{t_1}, \dots, S_{t_k} ($0 < t_1 < \dots < t_k < 1$) be the singular pages of the open book foliation $\mathcal{F}_{\text{ob}}(D_{\text{trans}})$ and $\varepsilon > 0$ be a sufficiently small number such that S_{t_i} is the only singular page in the interval $(t_i - \varepsilon, t_i + \varepsilon)$. Since D_{trans} is a transverse overtwisted disk with one negative elliptic point, by the definition of a transverse overtwisted disk [17, Definition 4.1], all the hyperbolic points of $\mathcal{F}_{\text{ob}}(D_{\text{trans}})$ are positive. This shows that $\pi(b_{t_i - \varepsilon}) \prec_{\text{right}} \pi(b_{t_i + \varepsilon})$ with $\text{Int}(\pi(b_{t_i - \varepsilon})) \cap \text{Int}(\pi(b_{t_i + \varepsilon})) = \emptyset$ for all $i = 1, \dots, k$ (see Figure 6 (ii), or consult Observation 2.5 of [21]). Let us put $\gamma_i = \pi(b_{t_i + \varepsilon}) = \pi(b_{t_{i+1} - \varepsilon}) \in \mathcal{A}_C(S; P)$. Then the sequence of arcs satisfies

$$\phi_L(\pi(b_1)) = \pi(b_0) = \gamma_0 \prec_{\text{right}} \gamma_1 \prec_{\text{right}} \dots \prec_{\text{right}} \gamma_k = \pi(b_1)$$

and $\text{Int}(\gamma_i) \cap \text{Int}(\gamma_{i+1}) = \emptyset$ hence $\phi_L \in \text{MCG}(S; P)$ is not quasi-right-veering. □

6. VERY POSITIVE FDTC AND NON-LOOSE LINKS

Proposition 3.12 and Theorem 1.2 show that a negative FDTC $c(\phi, L, C) < 0$ does not always imply looseness of the braid L . This makes a sharp contrast to empty braid case, where the negative FDTC $c(\phi, C) < 0$ implies overtwistedness of the contact structure $\xi_{(S, \phi)}$.

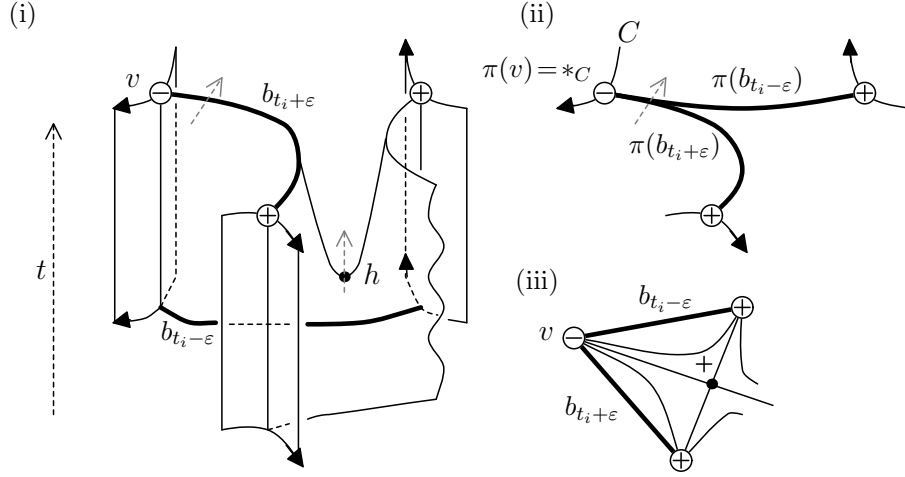


FIGURE 6. (i): A positive hyperbolic point h (saddle tangency). The gray dashed arrow indicate the positive normal vectors to the surface. Black arrows indicate the orientations of the binding components. (ii) Comparison of the b-arcs $\pi(b_{t_i-\varepsilon})$ and $\pi(b_{t_i+\varepsilon})$. (iii) Corresponding portion in the open book foliation $\mathcal{F}_{ob}(D_{\text{trans}})$.

On the other hand, if the FDTC is very positive then there is some similarity between non-empty braid case and empty braid case. In [21, Corollary 1.2] it is proved that a planar open book (S, ϕ) with $c(\phi, C) > 1$ for every boundary component C supports a tight contact structure. We may regard this as a special case ($L = \emptyset$) of the following theorem.

Theorem 6.1. *Let L be a closed braid with respect to a planar open book (S, ϕ) . If $c(L, \phi, C) > 1$ for every boundary component C of S then L is non-loose.*

Proof. By (2.2) the distinguished monodromy $\phi_L \in \mathcal{MCG}(S; P)$ gives

$$((S \setminus P) \times [0, 1]) / \sim_{\phi_L} \simeq M \setminus L.$$

Recall the forgetful map $f : \mathcal{MCG}(S; P) \rightarrow \mathcal{MCG}(S)$ in the Birman exact sequence (2.1). Note that $f(\phi_L) = \phi \in \mathcal{MCG}(S)$. In the following argument, we may use the open book (S, P, ϕ_L) instead of (S, ϕ) .

Assume that L is loose. By Theorem 5.1 there exists a transverse overtwisted disk D in $M \setminus L$. Applying the proof of [21, Theorem 1.1], we can construct a transverse overtwisted disk D' such that every b-arc of $\mathcal{F}_{ob}(D')$ ending at a valence ≤ 1 vertex of the graph $G_{--}(D')$ is an essential arc in the punctured page $S \setminus P$. Using [19, Lemma 5.7] the existence of such a disk D' implies that $c(\phi, L, C) = c(\phi_L, C) \leq 1$ for some boundary component C of S . \square

7. COMPARISON OF PROPOSED DEFINITIONS OF RIGHT-VEERINGNESS

In this section we discuss a comparison of several proposed definitions of right-veering for the mapping class group of punctured surfaces.

Definition 7.1. We say that an arc $\gamma : [0, 1] \rightarrow S$ is ∂ - P (resp. ∂ - ∂) arc if

- (1) $\gamma(0) \in \partial S$ and γ is transverse to ∂S at $\gamma(0)$.
- (2) $\gamma(t) \in \text{Int}(S) \setminus P$ for $t \in (0, 1)$.
- (3) $\gamma(1) \in P$ (resp. $\gamma(1) \in \partial S$ and γ is transverse to ∂S at $\gamma(1)$).
- (4) $\text{Int}(\gamma)$ is embedded in $S \setminus P$ and not boundary-parallel.

For a boundary component C of S , we say that a ∂ - P or ∂ - ∂ arc is *based on C* if $\gamma(0) \in C$.

As natural generalizations of the right-veering property for $MCG(S)$ to $MCG(S; P)$ there are three candidates.

Definition 7.2. For a boundary component C of S we say that $\psi \in MCG(S; P)$ is

- (1) ∂ -($\partial + P$) *right-veering with respect to C* if $\gamma \prec_{\text{right}} \psi(\gamma)$ or $\gamma = \psi(\gamma)$ for all ∂ - ∂ and ∂ - P arcs γ based on C .
- (2) ∂ - ∂ *right-veering with respect to C* if $\gamma \prec_{\text{right}} \psi(\gamma)$ or $\gamma = \psi(\gamma)$ for all ∂ - ∂ arcs γ based on C .
- (3) ∂ - P *right-veering with respect to C* if $\gamma \prec_{\text{right}} \psi(\gamma)$ or $\gamma = \psi(\gamma)$ for all ∂ - P arcs γ based on C .

We say that $\psi \in MCG(S; P)$ is ∂ -($\partial + P$), ∂ - ∂ , or, ∂ - P *right-veering*, respectively, if ψ is ∂ -($\partial + P$), ∂ - ∂ , or, ∂ - P right-veering, respectively, with respect to every boundary component of S .

The ∂ - ∂ right-veering appears in [3]. It is easy to see that our Definition 3.2 of right-veering is equivalent to the ∂ - ∂ right-veering. Recall that in Definition 3.2 we only consider ∂ - ∂ arcs starting from the distinguished base point $*_C \in C$. This restriction is just to define the orderings \prec_{right} and \ll_{right} on $\mathcal{A}_C(S; P)$.

On the other hand, in [2, 25] the notion of ∂ - P right-veering is used to study the classical braid group $MCG(D^2; P)$.

It is asked in [2, Remark 3.3] whether these two superficially different notions of “right-veering” are equivalent or not. One can immediately see that these notions (2) and (3) of “right-veering with respect to C ” are in general not exactly the same.

Example 7.3. Assume that S has more than one boundary components with marked points $P \neq \emptyset$. Let C and C' be distinct boundary components. Clearly $T_{C'}^{-1} \in MCG(S; P)$ is not ∂ - ∂ right-veering with respect to C . On the other hand $T_{C'}^{-1}$ preserves all ∂ - P arcs based on C . This means that $T_{C'}^{-1}$ is ∂ - P right-veering with respect to C .

More generally we have the following. Let $\psi \in MCG(S; P)$ be a ∂ - P right-veering map with respect to C . Suppose that $\psi(\gamma) = \gamma$ for some ∂ - ∂ arc γ connecting C and C' . Then $T_{C'}^{-1}\psi$ is still ∂ - P right-veering with respect to C , but is not ∂ - ∂ right-veering with respect to C since $T_{C'}^{-1}\psi(\gamma) = T_{C'}^{-1}(\gamma) \prec_{\text{right}} \gamma$.

It turns out that the difference between ∂ - ∂ right-veering and ∂ - P right-veering only shows up when $\psi \in MCG(S; P)$ involves negative Dehn twists along boundary components like in Example 7.3.

Definition 7.4. We say that $\psi \in MCG(S; P)$ is *special* with respect to C if the following are satisfied.

- ψ is not ∂ - ∂ right-veering with respect to C .
- If a ∂ - ∂ arc γ that is based on C and ending at C' has $\psi(\gamma) \prec_{\text{right}} \gamma$ then $C' \neq C$ and $\psi(\gamma) = T_{C'}^{-n}(\gamma)$ for some $n > 0$.

That is, a special map ψ is not ∂ - ∂ right-veering with respect to C only because of negative Dehn twists along other boundary components C' .

Theorem 7.5. Let $\psi \in MCG(S; P)$.

- (1) If ψ is ∂ - ∂ right-veering with respect to C , then ψ is ∂ - P right-veering with respect to C .
- (2) If ψ is ∂ - P right-veering with respect to C then either
 - ψ is ∂ - ∂ right-veering with respect to C , or,

- ψ is special with respect to C .

Proof. We prove both (1) and (2) by showing the contrapositives.

First we prove (1). Assume that there is a ∂ - P arc γ based on C with $\psi(\gamma) \prec_{\text{right}} \gamma$. Let κ be a properly embedded arc which is the boundary of a regular neighborhood of γ in S . Then κ yields a ∂ - ∂ arc with $\psi(\kappa) \prec_{\text{right}} \kappa$.

To see (2), assume that ψ is not ∂ - ∂ right-veering with respect to C and is not special with respect to C . Then there exists a ∂ - ∂ arc γ based on C such that $\psi(\gamma) \prec_{\text{right}} \gamma$. We put $\psi(\gamma)$ and γ so that they intersect efficiently. We will show that there exists a ∂ - P arc κ based on C with $\kappa(0) = \gamma(0)$ and $\psi(\gamma) \prec_{\text{right}} \kappa \prec_{\text{right}} \gamma$ if γ is not “bad”, which will be defined shortly. This shows $\psi(\kappa) \prec_{\text{right}} \psi(\gamma) \prec_{\text{right}} \kappa$, hence ψ cannot be ∂ - P right-veering with respect to C .

If $\#(\gamma, \psi(\gamma)) = m > 0$, we put $\text{Int}(\gamma) \cap \text{Int}(\psi(\gamma)) = \{p_1, \dots, p_m\} = \{q_1, \dots, q_m\}$, where $p_i = \gamma(t_i)$ with $0 < t_1 < t_2 < \dots < t_m < 1$ and $q_i = (\psi(\gamma))(s_i)$ with $0 < s_1 < s_2 < \dots < s_m < 1$. If $\#(\gamma, \psi(\gamma)) = m = 0$ we put $t_1 = s_1 = 1$ and $p_1 = q_1 = \gamma(1)$.

Suppose that $q_1 = p_k$. Let

$$\delta := \gamma|_{[0, t_k]} * (\psi(\gamma)|_{[0, s_1]})^{-1}$$

then δ is an oriented simple closed curve in $S \setminus P$. Here $*$ denotes the concatenation of arcs read from the left to the right, and $(-)^{-1}$ means the arc with reversed orientation. If δ is separating, we denote by R the connected component of $S \setminus (\delta \cup P)$ that lies on the left side of δ with respect to the orientation. If δ is non-separating let $R := S \setminus (\delta \cup P)$.

Definition 7.6. We say that the arc γ is *bad* if the following two properties are satisfied:

- R is an annulus (possibly pinched if $m = 0$) with no punctures. (In particular, δ is separating.)
- The sign of the intersection of γ and $\psi(\gamma)$ (in this order) at q_1 is positive.

Assume that γ is bad. Let $C' = \partial R \setminus \delta$. Note that C' is a boundary component of S . Since γ and $\psi(\gamma)$ intersect efficiently and δ is separating, $\psi(\gamma)$ cannot exit out of the annulus R . See Figure 7. Therefore we have:

Claim 7.7. If γ is bad then $C' \neq C$ and $\psi(\gamma) = T_{C'}^{-n}(\gamma)$ for some $n > 0$.

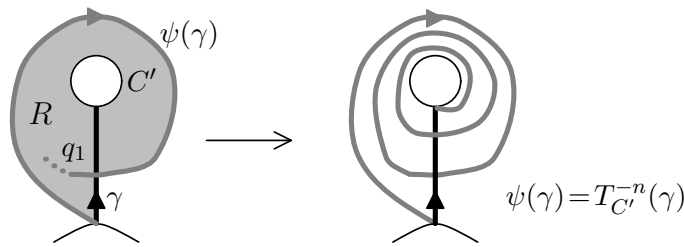


FIGURE 7. A bad arc γ and its image.

Since we assume that ψ is not special Claim 7.7 guarantees that γ is not bad.

Case 1: R is a punctured annulus or a non-annulus surface.

Take an arc γ' in $S \setminus (P \cup \gamma \cup \delta)$ which connects q_1 and a puncture point and efficiently intersects $\psi(\gamma)|_{[s_1, 1]}$.

Case 1A: γ' lies on the left side of γ near q_1 .

Take $\kappa := \gamma|_{[0, t_k]} * \gamma'$.

Case 1B: γ' lies on the right side of γ near q_1 .

If δ is separating and R is a punctured disk or a punctured annulus then let $\kappa \subset (R \setminus (R \cap \gamma))$ be an arc connecting $\gamma(0)$ and one of the punctures in R . We do not use γ' here.

Now we may assume that R is a planar surface with more than two boundary components or a surface with genus ≥ 1 . We can take an arc $\gamma'' \subset R$ connecting $\gamma(0)$ and q_1 such that:

- $\text{Int}(\gamma'')$ is disjoint from $\delta \cup \gamma \cup \gamma'$.
- γ'' is not parallel to δ .
- $\psi(\gamma) \prec_{\text{right}} \gamma'' \prec_{\text{right}} \gamma$.

Let $\kappa := \gamma'' * \gamma'$.

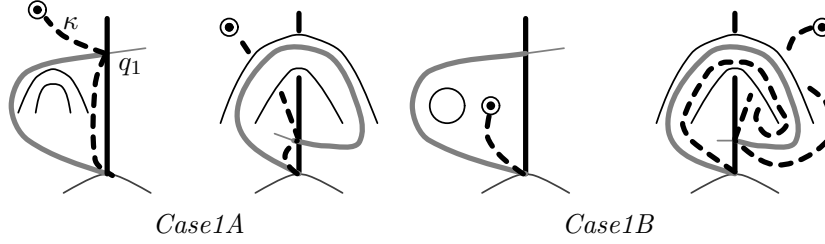


FIGURE 8. **Case 1.** A ∂ - P arc κ (dashed arc) is chosen so that it does not intersect γ (black bold line) and $\psi(\gamma)|_{[0,s_1]}$ (gray bold arc), possibly with one exceptional point q_1 .

Case 2: R is an annulus with no punctures.

Since γ is not bad the sign of the intersection of γ and $\psi(\gamma)$ at q_1 is negative. Let k' be the number satisfying $q_2 = p_{k'}$.

Case 2A: $k' < k$.

Since δ is separating the sign of the intersection of γ and $\psi(\gamma)$ at q_2 is positive. Take an arc γ' in $S \setminus (P \cup \gamma \cup \psi(\gamma)|_{[0,s_2]})$ which connects q_2 and a puncture point and efficiently intersects $\psi(\gamma)|_{[s_2,1]}$. Then put $\kappa := \psi(\gamma)|_{[0,s_1]} * (\gamma|_{[t_{k'},t_k]})^{-1} * \gamma'$.

Case 2B: $k < k'$.

Let γ' be an arc in $S \setminus (P \cup \gamma \cup \psi(\gamma)|_{[0,s_2]})$ that connects $\gamma(0)$ and a puncture point and put

$$\kappa := \begin{cases} \gamma|_{[0,t_{k'}]} * (\psi(\gamma)|_{[s_1,s_2]})^{-1} * (\gamma|_{[0,s_1]})^{-1} * \gamma' & (\text{if } \gamma \prec_{\text{right}} \gamma') \\ \gamma|_{[0,t_{k'}]} * (\psi(\gamma)|_{[s_1,s_2]})^{-1} * C * (\gamma|_{[0,s_1]})^{-1} * \gamma' & (\text{if } \gamma' \prec_{\text{right}} \psi(\gamma)) \end{cases}$$

In the second case in order to make κ embedded it turns along C . □

As a consequence of Theorem 7.5, the three notions of right-veering with respect to *all* the boundary components, which is a condition closely related to tight contact structures, are equivalent. In particular, if S has connected boundary then the three notions are equivalent.

Corollary 7.8. *For $\psi \in \mathcal{MCG}(S; P)$ the following are equivalent.*

- (1) ψ is ∂ -($\partial + P$) right-veering.
- (2) ψ is ∂ - ∂ right-veering.
- (3) ψ is ∂ - P right-veering.

Therefore in the case of $B_n = \mathcal{MCG}(D^2; \{n \text{ points}\})$ the proposed definitions of right-veering in [3] and [2, 25] are the same. Also, we remark that the subtle difference (special elements in Definition 7.4) only happens when $c(\psi, C) = 0$.

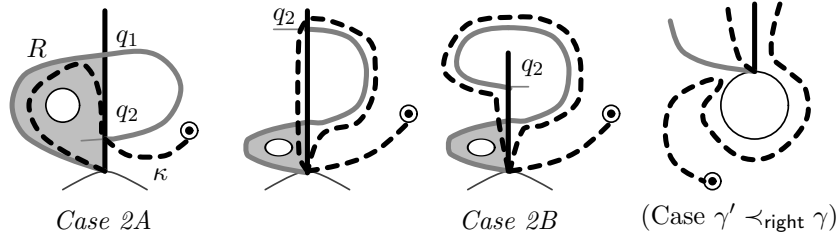


FIGURE 9. **Case 2.** Construction of a ∂ - P arc κ (dashed). κ does not intersect γ (black bold line) and $\psi(\gamma)|_{[0,s_2]}$ (gray bold arc), possibly with exceptions near q_1 , q_2 and $\gamma(1)$ (if $\gamma(1) \in C$).

Remark 7.9. One may come up with still different candidates of right-veering. Instead of using embedded arcs, one may use immersed arcs. However, one can check that immersed ∂ -($\partial + P$) (resp. ∂ - ∂ , ∂ - P) right-veering with respect to C is equivalent to the (embedded) ∂ -($\partial + P$) (resp. ∂ - ∂ , ∂ - P) right-veering with respect to C .

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN

E-mail address: `tetitoh@kurims.kyoto-u.ac.jp`

URL: `http://www.kurims.kyoto-u.ac.jp/~tetitoh/`

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IA 52242, USA

E-mail address: `keiko-kawamuro@uiowa.edu`